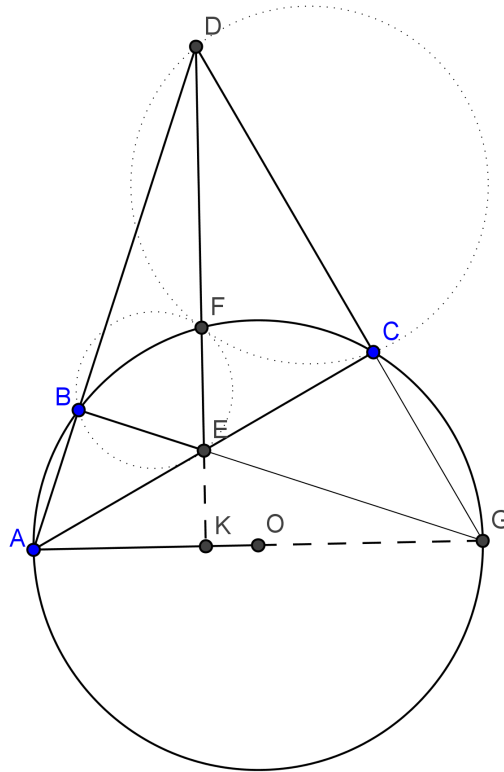


Problem 1.

Solution. Let $\ell \cap AO = \{K\}$ and G be the other end point of the diameter of Γ through A . Then D, C, G are collinear. Moreover, E is the orthocenter of triangle ADG . Therefore $GE \perp AD$ and G, E, B are collinear.



As $\angle CDF = \angle GDK = \angle GAC = \angle GFC$, FG is tangent to the circumcircle of triangle CFD at F . As $\angle FBE = \angle FBG = \angle FAG = \angle GFK = \angle GFE$, FG is also tangent to the circumcircle of BFE at F . Hence the circumcircles of the triangles CFD and BFE are tangent at F .



Problem 2.

Solution 1. We will obtain the inequality by adding the inequalities

$$(x+y)\sqrt{(z+x)(z+y)} \geq 2xy + yz + zx$$

for cyclic permutation of x, y, z .

Squaring both sides of this inequality we obtain

$$(x+y)^2(z+x)(z+y) \geq 4x^2y^2 + y^2z^2 + z^2x^2 + 4xy^2z + 4x^2yz + 2xyz^2$$

which is equivalent to

$$x^3y + xy^3 + z(x^3 + y^3) \geq 2x^2y^2 + xyz(x+y)$$

which can be rearranged to

$$(xy + yz + zx)(x - y)^2 \geq 0,$$

which is clearly true.

Solution 2. For positive real numbers x, y, z there exists a triangle with the side lengths $\sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x}$ and the area $K = \sqrt{xy + yz + zx}/2$.

The existence of the triangle is clear by simple checking of the triangle inequality. To prove the area formula, we have

$$K = \frac{1}{2}\sqrt{x+y}\sqrt{z+x}\sin\alpha,$$

where α is the angle between the sides of length $\sqrt{x+y}$ and $\sqrt{z+x}$. On the other hand, from the law of cosines we have

$$\cos\alpha = \frac{x+y+z+x-y-z}{2\sqrt{(x+y)(z+x)}} = \frac{x}{\sqrt{(x+y)(z+x)}}$$

and

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \frac{\sqrt{xy + yz + zx}}{\sqrt{(x+y)(z+x)}}.$$

Now the inequality is equivalent to

$$\sqrt{x+y}\sqrt{y+z}\sqrt{z+x} \sum_{cyc} \sqrt{x+y} \geq 16K^2.$$

This can be rewritten as

$$\frac{\sqrt{x+y}\sqrt{y+z}\sqrt{z+x}}{4K} \geq 2 \frac{K}{\sum_{cyc} \sqrt{x+y}/2}$$

to become the Euler inequality $R \geq 2r$.



Problem 3.

Solution 1. Let $\alpha = 3/2$ so $1 + \alpha > \alpha^2$.

Given y , we construct Y algorithmically. Let $Y = \emptyset$ and of course $S_\emptyset = 0$. For $i = 0$ to m , perform the following operation:

$$\text{If } S_Y + 2^i 3^{m-i} \leq y, \text{ then replace } Y \text{ by } Y \cup \{2^i 3^{m-i}\}.$$

When this process is finished, we have a subset Y of P_m such that $S_Y \leq y$.

Notice that the elements of P_m are in ascending order of size as given, and may alternatively be described as $2^m, 2^m \alpha, 2^m \alpha^2, \dots, 2^m \alpha^m$. If any member of this list is not in Y , then no two consecutive members of the list to the left of the omitted member can both be in Y . This is because $1 + \alpha > \alpha^2$, and the greedy nature of the process used to construct Y .

Therefore either $Y = P_m$, in which case $y = 3^{m+1} - 2^{m+1}$ and all is well, or at least one of the two leftmost elements of the list is omitted from Y .

If 2^m is not omitted from Y , then the algorithmic process ensures that $(S_Y - 2^m) + 2^{m-1} 3 > y$, and so $y - S_Y < 2^m$. On the other hand, if 2^m is omitted from Y , then $y - S_Y < 2^m$.

Solution 2. Note that $3^{m+1} - 2^{m+1} = (3 - 2)(3^m + 3^{m-1} \cdot 2 + \dots + 3 \cdot 2^{m-1} + 2^m) = S_{P_m}$. Dividing every element of P_m by 2^m gives us the following equivalent problem:

Let m be a positive integer, $a = 3/2$, and $Q_m = \{1, a, a^2, \dots, a^m\}$. Show that for any real number x satisfying $0 \leq x \leq 1 + a + a^2 + \dots + a^m$, there exists a subset X of Q_m such that $0 \leq x - S_X < 1$.

We will prove this problem by induction on m . When $m = 1$, $S_\emptyset = 0$, $S_{\{1\}} = 1$, $S_{\{a\}} = 3/2$, $S_{\{1,a\}} = 5/2$. Since the difference between any two consecutive of them is at most 1, the claim is true.

Suppose that the statement is true for positive integer m . Let x be a real number with $0 \leq x \leq 1 + a + a^2 + \dots + a^{m+1}$. If $0 \leq x \leq 1 + a + a^2 + \dots + a^m$, then by the induction hypothesis there exists a subset X of $Q_m \subset Q_{m+1}$ such that $0 \leq x - S_X < 1$.

If $\frac{a^{m+1} - 1}{a - 1} = 1 + a + a^2 + \dots + a^m < x$, then $x > a^{m+1}$ as

$$\frac{a^{m+1} - 1}{a - 1} = 2(a^{m+1} - 1) = a^{m+1} + (a^{m+1} - 2) \geq a^{m+1} + a^2 - 2 = a^{m+1} + \frac{1}{4}.$$

Therefore $0 < (x - a^{m+1}) \leq 1 + a + a^2 + \dots + a^m$. Again by the induction hypothesis, there exists a subset X of Q_m satisfying $0 \leq (x - a^{m+1}) - S_X < 1$. Hence $0 \leq x - S_{X'} < 1$ where $X' = X \cup \{a^{m+1}\} \subset Q_{m+1}$.



Problem 4.

Solution 1. There are three such functions: the constant functions 1, 2 and the identity function $\text{id}_{\mathbf{Z}^+}$. These functions clearly satisfy the conditions in the hypothesis. Let us prove that there are only ones.

Consider such a function f and suppose that it has a fixed point $a \geq 3$, that is $f(a) = a$. Then $a!, (a!)!, \dots$ are all fixed points of f , hence the function f has a strictly increasing sequence $a_1 < a_2 < \dots < a_k < \dots$ of fixed points. For a positive integer n , $a_k - n$ divides $a_k - f(n) = f(a_k) - f(n)$ for every $k \in \mathbf{Z}^+$. Also $a_k - n$ divides $a_k - n$, so it divides $a_k - f(n) - (a_k - n) = n - f(n)$. This is possible only if $f(n) = n$, hence in this case we get $f = \text{id}_{\mathbf{Z}^+}$.

Now suppose that f has no fixed points greater than 2. Let $p \geq 5$ be a prime and notice that by Wilson's Theorem we have $(p-2)! \equiv 1 \pmod{p}$. Therefore p divides $(p-2)! - 1$. But $(p-2)! - 1$ divides $f((p-2)!) - f(1)$, hence p divides $f((p-2)!) - f(1) = (f(p-2))! - f(1)$. Clearly we have $f(1) = 1$ or $f(1) = 2$. As $p \geq 5$, the fact that p divides $(f(p-2))! - f(1)$ implies that $f(p-2) < p$. It is easy to check, again by Wilson's Theorem, that p does not divide $(p-1)! - 1$ and $(p-1)! - 2$, hence we deduce that $f(p-2) \leq p-2$. On the other hand, $p-3 = (p-2) - 1$ divides $f(p-2) - f(1) \leq (p-2) - 1$. Thus either $f(p-2) = f(1)$ or $f(p-2) = p-2$. As $p-2 \geq 3$, the last case is excluded, since the function f has no fixed points greater than 2. It follows $f(p-2) = f(1)$ and this property holds for all primes $p \geq 5$. Taking n any positive integer, we deduce that $p-2-n$ divides $f(p-2) - f(n) = f(1) - f(n)$ for all primes $p \geq 5$. Thus $f(n) = f(1)$, hence f is the constant function 1 or 2.

Solution 2. Note first that if $f(n_0) = n_0$, then $m - n_0 | f(m) - m$ for all $m \in \mathbf{Z}^+$. If $f(n_0) = n_0$ for infinitely many $n_0 \in \mathbf{Z}^+$, then $f(m) - m$ has infinitely many divisors, hence $f(m) = m$ for all $m \in \mathbf{Z}^+$. On the other hand, if $f(n_0) = n_0$ for some $n_0 \geq 3$, then f fixes each term of the sequence $(n_k)_{k=0}^{\infty}$, which is recursively defined by $n_k = n_{k-1}!$. Hence if $f(3) = 3$, then $f(n) = n$ for all $n \in \mathbf{Z}^+$.

We may assume that $f(3) \neq 3$. Since $f(1) = f(1)!$, and $f(2) = f(2)!$, $f(1), f(2) \in \{1, 2\}$. We have $4 = 3! - 2 | f(3)! - f(2)$. This together with $f(3) \neq 3$ implies that $f(3) \in \{1, 2\}$. Let $n > 3$, then $n! - 3 | f(n)! - f(3)$ and $3 \nmid f(n)!$, i.e. $f(n)! \in \{1, 2\}$. Hence we conclude that $f(n) \in \{1, 2\}$ for all $n \in \mathbf{Z}^+$.

If f is not constant, then there exist positive integers m, n with $\{f(n), f(m)\} = \{1, 2\}$. Let $k = 2 + \max\{m, n\}$. If $f(k) \neq f(m)$, then $k - m | f(k) - f(m)$. This is a contradiction as $|f(k) - f(m)| = 1$ and $k - m \geq 2$.

Therefore the functions satisfying the conditions are $f \equiv 1, f \equiv 2, f = \text{id}_{\mathbf{Z}^+}$.

