Inequalities Marathon

Problem 1 *'India 2002'* (Hassan Al-Sibyani): For any positive real numbers a, b, c show that the following inequality holds

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{c+a}{c+b}+\frac{a+b}{a+c}+\frac{b+c}{b+a}$$

First Solution (Popa Alexandru): Ok. After not so many computations i got that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\frac{a+b}{c+a}-\frac{b+c}{a+b}-\frac{c+a}{b+c}$$

$$=\frac{abc}{(a+b)(b+c)(c+a)}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-\frac{a}{b}-\frac{b}{c}-\frac{c}{a}\right)$$

$$+\frac{abc}{(a+b)(b+c)(c+a)}\left(\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ca}{b^2}-3\right)$$

So in order to prove the above inequality we need to prove $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ and $\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ca}{b^2} \geq 3$ The second inequality is obvious by AM-GM , and the for the first we have:

$$\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2 \geq 3\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right) \geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2$$

where i used AM-GM and the inequality $3(x^2+y^2+z^2) \geq (x+y+z)^2$ for $x=\frac{a}{b}$, $y=\frac{b}{c}$, $z=\frac{c}{a}$
So the inequality is proved.

Second Solution (Raghav Grover): Substitute $\frac{a}{b}=x, \frac{b}{c}=y, \frac{c}{a}=z$ So $xyz=1$. The inequality after substitution becomes
$x^2z+y^2x+z^2x+x^2+y^2+z^2 \geq x+y+z+3$
$x^2z+y^2x+z^2x \geq 3$ So now it is left to prove that $x^2+y^2+z^2 \geq x+y+z$ which is easy.

Third Solution (Popa Alexandru): Bashing out it gives

$$a^4c^2+b^4a^2+c^4b^2+a^3b^3+b^3c^3+a^3c^3 \geq abc(ab^2+bc^2+ca^2+3abc)$$

which is true because AM-GM gives :

$$a^3b^3+b^3c^3+a^3c^3 \geq 3a^2b^2c^2$$

and by Muirhead :

$$a^4c^2+b^4a^2+c^4b^2 \geq abc(ab^2++bc^2+ca^2)$$

Fourth Solution (Popa Alexandru): Observe that the inequality is equivalent with:

$$\sum_{cyc}\frac{a^2+bc}{a(a+b)} \geq 3$$

Now use AM-GM:

$$\sum_{cyc}\frac{a^2+bc}{a(a+b)} \geq 3\sqrt[3]{\frac{\prod(a^2+bc)}{abc\prod(a+b)}}$$

So it remains to prove:

$$\prod(a^2+bc) \geq abc\prod(a+b)$$

Now we prove

$$(a^2+bc)(b^2+ca) \geq ab(c+a)(b+c) \Leftrightarrow a^3+b^3 \geq ab^2+a^2b \Leftrightarrow (a+b)(a-b)^2 \geq 0$$

Multiplying the similars we are done.

Problem 2 *'Maxim Bogdan'* (Popa Alexandru): Let $a, b, c, d > 0$ such that $a \leq b \leq c \leq d$ and $abcd = 1$. Then show that:

$$(a+1)(d+1) \geq 3 + \frac{3}{4d^3}$$

First Solution (Mateescu Constantin):From the condition $a \leq b \leq c \leq d$ we get that $a \geq \frac{1}{d^3}$.

$\Longrightarrow\ (a+1)(d+1) \geq \left(\frac{1}{d^3}+1\right)(d+1)$

Now let's prove that $\left(1+\frac{1}{d^3}\right)(d+1) \geq 3+\frac{3}{4d^3}$

This is equivalent with: $(d^3+1)(d+1) \geq 3d^3+\frac{3}{4}$

$\Longleftrightarrow [d(d-1)]^2-[d(d-1)]+1 \geq \frac{3}{4} \Longleftrightarrow \left[d(d-1)-\frac{1}{2}\right]^2 \geq 0.$

Equality holds for $a=\frac{1}{d^3}$ and $d(d-1)-\frac{1}{2}=0 \Longleftrightarrow d=\frac{1+\sqrt{3}}{2}$

Problem 3 *'Darij Grinberg'* (Hassan Al-Sibyani): If a, b, c are three positive real numbers, then

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}$$

First Solution (Dimitris X):

$\sum \frac{a^2}{ab^2+ac^2+2abc} \geq \frac{(a+b+c)^2}{\sum_{sym} a^2b+6abc}$

So we only have to prove that:

$4(a+b+c)^3 \ \geq\ 9\sum_{sym} a^2b+54abc \Longleftrightarrow 4(a^3+b^3+c^3)+12\sum_{sym} a^2b+24abc \geq 9\sum_{sym} a^2b+54abc \Longleftrightarrow$

$4(a^3+b^3+c^3)+3\sum_{sym} a^2b \geq 30abc$

But $\sum_{sym} a^2b \geq 6abc$ and $a^3+b^3+c^3 \geq 3abc$

So $4(a^3+b^3+c^3)+3\sum_{sym} a^2b \geq 30abc$

Second Solution (Popa Alexandru): Use Cauchy-Schwartz and Nesbitt:

$$(a+b+c)\left(\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right) \geq \left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2 \geq \frac{9}{4}$$

Problem 4 *'United Kingdom'* (Dimitris X): For $a, b, c \geq 0$ and $a+b+c = 1$ prove that $7(ab+bc+ca) \leq 2+9abc$

First Solution (Popa Alexandru):
Homogenize to

$$2(a+b+c)^3 + 9abc \geq 7(ab+bc+ca)(a+b+c)$$

Expanding it becomes :

$$\sum_{sym} a^3 + 6\sum_{sym} a^2b + 21abc \geq 7\sum_{sym} a^2b + 21abc$$

So we just need to show:

$$\sum_{sym} a^3 \geq \sum_{sym} a^2b$$

which is obvious by
$a^3 + a^3 + b^3 \geq 3a^2b$ and similars.

Second Solution (Popa Alexandru): Schur gives $1 + 9abc \geq 4(ab + bc + ca)$ and use also $3(ab + bc + ca) \leq (a + b + c)^2 = 1$ Suming is done .

Problem 5 *'Gheorghe Szollosy, Gazeta Matematica'* (Popa Alexandru): Let $x, y, z \in \mathbb{R}_+$. Prove that:

$$\sqrt{x(y+1)} + \sqrt{y(z+1)} + \sqrt{z(x+1)} \leq \frac{3}{2}\sqrt{(x+1)(y+1)(z+1)}$$

First Solution (Endrit Fejzullahu): Dividing with the square root on the RHS we have :

$$\sqrt{\frac{x}{(x+1)(z+1)}} + \sqrt{\frac{y}{(x+1)(y+1)}} + \sqrt{\frac{z}{(y+1)(z+1)}} \leq \frac{3}{2}$$

By AM-GM

$$\sqrt{\frac{x}{(x+1)(z+1)}} \leq \frac{1}{2}\left(\frac{x}{x+1} + \frac{1}{y+1}\right)$$
$$\sqrt{\frac{y}{(x+1)(y+1)}} \leq \frac{1}{2}\left(\frac{y}{y+1} + \frac{1}{x+1}\right)$$
$$\sqrt{\frac{z}{(y+1)(z+1)}} \leq \frac{1}{2}\left(\frac{z}{z+1} + \frac{1}{y+1}\right)$$

Summing we obtain

$$LHS \leq \frac{1}{2}\left(\left(\frac{x}{x+1} + \frac{1}{x+1}\right) + \left(\frac{y}{y+1} + \frac{1}{y+1}\right) + \left(\frac{z}{z+1} + \frac{1}{z+1}\right)\right) = \frac{3}{2}$$

Problem 6 *'—'* (Endrit Fejzullahu): Let a, b, c be positive numbers , then prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{4a}{2a^2+b^2+c^2} + \frac{4b}{a^2+2b^2+c^2} + \frac{4c}{a^2+b^2+2c^2}$$

First Solution (Mateescu Constantin): By $AM - GM$ we have $2a^2 + b^2 + c^2 \geq 4a\sqrt{bc}$
$\Longrightarrow \dfrac{4a}{2a^2 + b^2 + c^2} \leq \dfrac{4a}{4a\sqrt{bc}} = \dfrac{1}{\sqrt{bc}}$
Addind the similar inequalities $\Longrightarrow RHS \leq \dfrac{1}{\sqrt{ab}} + \dfrac{1}{\sqrt{bc}} + \dfrac{1}{\sqrt{ca}}$ (1)
Using Cauchy-Schwarz we have $\left(\dfrac{1}{\sqrt{ab}} + \dfrac{1}{\sqrt{bc}} + \dfrac{1}{\sqrt{ca}}\right)^2 \leq \left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)^2$
So $\dfrac{1}{\sqrt{ab}} + \dfrac{1}{\sqrt{bc}} + \dfrac{1}{\sqrt{ca}} \leq \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$ (2)
From (1), (2) we obtain the desired result .

Second Solution (Popa Alexandru): By Cauchy-Schwatz :

$$\frac{4a}{2a^2 + b^2 + c^2} \leq \frac{a}{a^2 + b^2} + \frac{a}{a^2 + c^2}$$

Then we have

$$RHS \leq \sum_{cyc} \frac{a+b}{a^2 + b^2} \leq \sum_{cyc} \frac{2}{a+b} \leq \sum_{cyc} \left(\frac{1}{2a} + \frac{1}{2b}\right) = LHS$$

Problem 7 '—' (Mateescu Constantin): Let a, b, c, d, e be non-negative real numbers such that $a + b + c + d + e = 5$. Prove that:

$$abc + bcd + cde + dea + eab \leq 5$$

First Solution (Popa Alexandru): Assume $e \leq \min\{a, b, c, d\}$. Then AM-GM gives :

$$e(c+a)(b+d) + bc(a+d-e) \leq \frac{e(5-e)^2}{4} + \frac{(5-2e)^2}{27} \leq 5$$

the last one being equivalent with:

$$(e-1)^2(e+8) \geq 0$$

Problem 8 *'Popa Alexandru'* (Popa Alexandru): Let a, b, c be real numbers such that $0 \leq a \leq b \leq c$. Prove that:

$$(a+b)(c+a)^2 \geq 6abc$$

First Solution (Popa Alexandru): Let

$$b = xa\ ,\ c = yb = xya \Rightarrow x, y \geq 1$$

Then:

$$\frac{(a+b)(a+c)^2}{3} \geq 2abc$$

$$\Leftrightarrow (x+1)(xy+1)^2 \cdot a^3 \geq 6x^2ya^3$$

$$\Leftrightarrow (x+1)(xy+1)^2 \geq 6x^2y$$
$$\Leftrightarrow (x+1)(4xy+(xy-1)^2) \geq 6x^2y$$
$$\Leftrightarrow 4xy+(xy-1)^2 \cdot x+(xy-1)^2-2x^2y \geq 0$$

We have that:

$$4xy+(xy-1)^2 \cdot x+(xy-1)^2-2x^2y \geq$$
$$\geq 4xy+2(xy-1)^2-2x^2y(\text{ because } x \geq 1)$$
$$= 2x^2y^2+2-2x^2y = 2xy(y-1)+2 > 0$$

done.

Second Solution (Endrit Fejzullahu): Let $b = a+x, c = b+y = a+x+y$,sure $x, y \geq 0$
Inequality becomes

$$(2a+x)(x+y+2a)^2-6a(a+x)(a+x+y) \geq 0$$

But

$$(2a+x)(x+y+2a)^2-6a(a+x)(a+x+y) = 2a^3+2a^2y+2axy+2ay^2+x^3+2x^2y+xy^2$$

which is clearly positive.

Problem 9 '—' (Raghav Grover): Prove for positive reals

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq 2$$

First Solution (Dimitris X):
From andreescu $LHS \geq \dfrac{(a+b+c+d)^2}{\sum\limits_{sym} ab+(ac+bd)}$
So we only need to prove that:

$$(a+b+c+d)^2 \geq 2\sum_{sym} ab+2(ac+bd) \Longleftrightarrow (a-c)^2+(b-d)^2 \geq 0....$$

Problem 10 '—' (Dimitris X): Let a, b, c, d be REAL numbers such that $a^2+b^2+c^2+d^2 = 4$ Prove that:

$$a^3+b^3+c^3+d^3 \leq 8$$

First Solution (Popa Alexandru): Just observe that

$$a^3+b^3+c^3+d^3 \leq 2(a^2+b^2+c^2+d^2) = 8$$

because $a, b, c, d \leq 2$

Problem 11 *'—'* (Endrit Fejzullahu): Let a, b, c be positive real numbers such that $abc = 1$ .Prove that

$$\sum_{cyc} \frac{1}{a^2 + 2b^2 + 3} \leq \frac{1}{2}$$

First Solution (Popa Alexandru): Using AM-GM we have :

$$LHS = \sum_{cyc} \frac{1}{(a^2 + b^2) + (b^2 + 1) + 2} \leq \sum_{cyc} \frac{1}{2ab + 2b + 2}$$

$$= \frac{1}{2} \sum_{cyc} \frac{1}{ab + b + 1} = \frac{1}{2}$$

because

$$\frac{1}{bc + c + 1} = \frac{1}{bc + c + abc} = \frac{1}{c} \cdot \frac{1}{ab + b + 1} = \frac{ab}{ab + b + 1}$$

and

$$\frac{1}{ca + a + 1} = \frac{1}{\frac{1}{b} + a + 1} = \frac{b}{ab + b + 1}$$

so

$$\sum_{cyc} \frac{1}{ab + b + 1} = \frac{1}{ab + b + 1} + \frac{ab}{ab + b + 1} + \frac{b}{ab + b + 1} = 1$$

Problem 12 *'Popa Alexandru'* (Popa Alexandru): Let $a, b, c > 0$ such that $a + b + c = 1$. Prove that:

$$\frac{1 + a + b}{2 + c} + \frac{1 + b + c}{2 + a} + \frac{1 + c + a}{2 + b} \geq \frac{15}{7}$$

First Solution (Dimitris X):
$\sum \frac{1 + a + b}{2 + c} + 1 \geq \frac{15}{7} + 3 \Longleftrightarrow \sum \frac{3 + (a + b + c)}{2 + c} \geq \frac{36}{7} \Longleftrightarrow \sum \frac{4}{2 + c} \geq \frac{36}{7}$
But $\sum \frac{2^2}{2 + c} \geq \frac{(2 + 2 + 2)^2}{2 + 2 + 2 + a + b + c} = \frac{36}{7}$

Second Solution (Endrit Fejzullahu): Let $a \geq b \geq c$ then by Chebyshev's inequality we have
$LHS \geq \frac{1}{3}(1 + 1 + 1 + 2(a + b + c)) \sum_{cyc} \frac{1}{2 + a} = \frac{5}{3} \sum_{cyc} \frac{1}{2 + a}$
By Titu's Lemma $\sum_{cyc} \frac{1}{2 + a} \geq \frac{9}{7}$, then $LHS \geq \frac{15}{7}$

Problem 13 *'Titu Andreescu, IMO 2000'* (Dimitris X): Let a, b, c be positive so that $abc = 1$

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

First Solution (Endrit Fejzullahu):
$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1$
Substitute $a=\frac{x}{y}, b=\frac{y}{z}$
Inequality is equivalent with
$\left(\frac{x}{y}-1+\frac{z}{y}\right)\left(\frac{y}{z}-1+\frac{x}{z}\right)\left(\frac{z}{x}-1+\frac{y}{x}\right) \leq 1$
$\iff (x+z-y)(y-z+x)(z-x+y) \leq xyz$
WLOG ,Let $x>y>z$, then $x+z>y, x+y>z$.If $y+z<x$, then we are done because
$(x+z-y)(y-z+x)(z-x+y) \leq 0$ and $xyz \geq 0$
Otherwise if $y+z>x$, then x,y,z are side lengths of a triangle ,and then we can make the substitution
$x=m+n, y=n+t$ and $z=t+m$
Inequality is equivalent with
$8mnt \leq (m+n)(n+t)(t+m)$, this is true by AM-GM
$m+n \geq 2\sqrt{mn}, n+t \geq 2\sqrt{nt}$ and $t+m \geq 2\sqrt{tm}$, multiply and we're done.

Problem 14 *'Korea 1998'* (Endrit Fejzullahu): Let $a,b,c>0$ and $a+b+c=abc$. Prove that:

$$\frac{1}{\sqrt{a^2+1}}+\frac{1}{\sqrt{b^2+1}}+\frac{1}{\sqrt{c^2+1}} \leq \frac{3}{2}$$

First Solution (Dimitris X): Setting $a=\frac{1}{x}, b=\frac{1}{y}, c=\frac{1}{z}$ the condition becomes $xy+yz+zx=1$, and the inequality:
$\sum \frac{x}{\sqrt{x^2+1}} \leq \frac{3}{2}$
But $\sum \frac{x}{\sqrt{x^2+1}} = \sum \frac{x}{\sqrt{x^2+xy+xz+zy}} = \sum \sqrt{\frac{x}{x+y}\frac{x}{x+z}}$
But $\sqrt{\frac{x}{x+y}\frac{x}{x+z}} \leq \frac{\frac{x}{x+y}+\frac{x}{x+z}}{2}$
So $\sum \sqrt{\frac{x}{x+y}\frac{x}{x+z}} \leq \frac{\frac{x}{x+y}+\frac{y}{x+y}+\frac{x}{z+x}+\frac{z}{z+x}+\frac{y}{y+z}+\frac{z}{z+y}}{2} = \frac{3}{2}$
Second Solution (Raghav Grover):
Substitute $a=\tan x$,$b=\tan y$ and $c=\tan z$ where $x+y+z=\pi$
And we are left to prove
$\cos x+\cos y+\cos z \leq \frac{3}{2}$
Which i think is very well known..

Third Solution (Endrit Fejzullahu): By AM-GM we have $a+b+c \geq 3\sqrt[3]{abc}$ and since $a+b+c=abc \implies (abc)^2 \geq 27$
We rewrite the given inequality as
$\frac{1}{3}\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right) \leq \frac{1}{2}$

Since function $f(a)=\dfrac{1}{\sqrt{a^2+1}}$ is concave ,we apply Jensen's inequality
$\frac{1}{3}f(a)+\frac{1}{3}f(b)+\frac{1}{3}f(c)\le f\left(\frac{a+b+c}{3}\right)=f\left(\frac{abc}{3}\right)=\frac{1}{\sqrt{\frac{(abc)^2}{3^2}+1}}\le\frac{1}{2}\iff(abc)^2\ge 27,\mathbb{QED}$

Problem 15 '—' (Dimitris X):
If $a,b,c\in\mathbb{R}$ and $a^2+b^2+c^2=3$. Find the minimum value of $A=ab+bc+ca-3(a+b+c)$.

First Solution (Endrit Fejzullahu):
$ab+bc+ca=\frac{(a+b+c)^2-a^2-b^2-c^2}{2}=\frac{(a+b+c)^2-3}{2}$
Let $a+b+c=x$
Then $A=\frac{x^2}{2}-3x-\frac{3}{2}$
We consider the second degree fuction $f(x)=\frac{x^2}{2}-3x-\frac{3}{2}$
We obtain minimum for $f\left(\frac{-b}{2a}\right)=f(3)=-6$
Then $A_{min}=-6$,it is attained for $a=b=c=1$

Problem 16 '—' (Endrit Fejzullahu): If a,b,c are positive real numbers such that $a+b+c=1$.Prove that

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\ge\sqrt{\frac{3}{2}}$$

First Solution (keyree10): Let $f(x)=\frac{x}{\sqrt{1-x}}$. $f"(x)>0$
Therefore, $\sum\frac{a}{\sqrt{1-a}}\ge\frac{3s}{\sqrt{1-s}}$,where $s=\frac{a+b+c}{3}=\frac{1}{3}$, by jensen's.
$\implies\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\ge\sqrt{\frac{3}{2}}$. Hence proved
Second Solution (geniusbliss):
By holders' inequality,
$\left(\sum_{cyclic}\frac{a}{(b+c)^{\frac{1}{2}}}\right)\left(\sum_{cyclic}\frac{a}{(b+c)^{\frac{1}{2}}}\right)\left(\sum_{cyclic}a(b+c)\right)\ge(a+b+c)^3$
thus,
$\left(\sum_{cyclic}\frac{a}{(b+c)^{\frac{1}{2}}}\right)^2\ge\frac{(a+b+c)^2}{2(ab+bc+ca)}\ge\frac{3(ab+bc+ca)}{2(ab+bc+ca)}=\frac{3}{2}$
or,
$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\ge\sqrt{\frac{3}{2}}$
Third Solution (Redwane Khyaoui):

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\ge\frac{1}{3}(a+b+c)\left(\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}}+\frac{1}{\sqrt{a+b}}\right)$$

$f(x) = \frac{1}{\sqrt{x}}$ is a convex function, so jensen's inequality gives:

$$LHS \geq \frac{1}{3}\left(3 \cdot \frac{1}{\sqrt{\frac{2}{3}(a+b+c)}}\right) = \sqrt{\frac{3}{2}}$$

Problem 17 *'—'* (keyree10): If a, b, c are REALS such that $a^2 + b^2 + c^2 = 1$ Prove that

$$a + b + c - 2abc \leq \sqrt{2}$$

First Solution (Popa Alexandru):
Use Cauchy-Schwartz:

$$LHS = a(1 - 2bc) + (b + c) \leq \sqrt{(a^2 + (b+c)^2)((1-2bc)^2 + 1)}$$

So it'll be enough to prove that :

$$(a^2 + (b+c)^2)((1-2bc)^2 + 1) \leq 2 \Leftrightarrow (1 + 2bc)(1 - bc + 2b^2c^2) \leq 1 \Leftrightarrow 4b^2c^2 \leq 1$$

which is true because

$$1 \geq b^2 + c^2 \geq 2bc$$

done

Problem 18 *'—'* (Popa Alexandru): Let $x, y, z > 0$ such that $xyz = 1$. Show that:

$$x^2 + y^2 + z^2 + x + y + z \geq 2(xy + yz + zx)$$

First Solution (great math): To solve the problem of alex, we need Schur and Cauchy inequality as demonstrated as follow

$$a + b + c \geq 3\sqrt[3]{abc} = \sqrt[3]{(abc)^2} \geq \frac{9abc}{a+b+c} \geq 2(ab + bc + ca) - (a^2 + b^2 + c^2)$$

Note that we possess the another form of Schur such as

$$(a^2 + b^2 + c^2)(a + b + c) + 9abc \geq 2(ab + bc + ca)(a + b + c)$$

Therefore, needless to say, we complete our proof here.

Problem 19 *'Hoang Quoc Viet'* (Hoang Quoc Viet): Let a, b, c be positive reals satisfying $a^2 + b^2 + c^2 = 3$. Prove that

$$(abc)^2(a^3 + b^3 + c^3) \leq 3$$

First Solution (Hoang Quoc Viet): Let

$$A = (abc)^2(a^3 + b^3 + c^3)$$

Therefore, we only need to maximize the following expression

$$A^3 = (abc)^6(a^3 + b^3 + c^3)^3$$

Using Cauchy inequality as follows, we get

$$A^3 = \frac{1}{3^6}\left(3a^2b\right)\left(3a^2c\right)\left(3b^2a\right)\left(3b^2c\right)\left(3c^2a\right)\left(3c^2b\right)\left(a^3 + b^3 + c^3\right)^3 \le \left(\frac{3(a^2 + b^2 + c^2)(a + b + c)}{9}\right)^9$$

It is fairly straightforward that

$$a + b + c \le \sqrt{3(a^2 + b^2 + c^2)} = 3$$

Therefore,

$$A^3 \le 3^3$$

which leads to $A \le 3$ as desired. The equality case happens $\iff a = b = c = 1$

Second Solution (FantasyLover):
For the sake of convenience, let us introduce the new unknowns u, v, w as follows:

$$\begin{aligned} u &= a + b + c \\ v &= ab + bc + ca \\ w &= abc \end{aligned}$$

Now note that $u^2 - 2v = 3$ and $a^3 + b^3 + c^3 = u(u^2 - 3v) = u\left(\frac{9 - u^2}{2}\right)$.
We are to prove that $w^2\left(u \cdot \frac{9 - u^2}{2} + 3w\right) \le 3$.
By AM-GM, we have $\sqrt[3]{abc} \le \frac{a + b + c}{3} \implies w \le \frac{u^3}{3^3}$.
Hence, it suffices to prove that $u^7 \cdot \frac{9 - u^2}{2} + \frac{u^9}{3^2} \le 3^7$.
However, by QM-AM we have $\sqrt{\frac{a^2 + b^2 + c^2}{3}} \ge \frac{a + b + c}{3} \implies u \le 3$
differentiating, u achieves its maximum when $\frac{7u^6(9 - u^2)}{2} = 0$.
Since a, b, c are positive, u cannot be 0, and the only possible value for u is 3.
Since $u \le 3$, the above inequality is true.

Problem 20 *'Murray Klamkin, IMO 1983'* (Hassan Al-Sibyani): Let a, b, c be the lengths of the sides of a triangle. Prove that:

$$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \ge 0$$

First Solution (Popa Alexandru): Use Ravi substitution $a = x + y$, $b = y + z$, $c = z + x$ then the inequality becomes :

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z \ ,$$

true by Cauchy-Schwartz.

Second Solution (geniusbliss): we know from triangle inequality that $b \geq (a - c)$ and $c \geq (b - a)$ and $a \geq (c - b)$
therefore,
$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq a^2(a-c)(a-b) + b^2(b-a)(b-c) + c^2(c-b)(c-a) \geq 0$
and the last one is schur's inequality for $r = 2$ so proved with the equality holding when $a = b = c$ or for and equilateral triangle

Problem 21 *'Popa Alexandru'* (Popa Alexandru): Let $x, y, z \in \left[\frac{1}{3}, \frac{2}{3}\right]$. Show that :

$$1 \geq \sqrt[3]{xyz} + \frac{2}{3(x+y+z)}$$

First Solution (Endrit Fejzullahu):
By Am-Gm $x + y + z \geq 3\sqrt[3]{xyz}$,then
$\frac{2}{3(x+y+z)} + \sqrt[3]{xyz} \leq \frac{2}{9}\sqrt[3]{xyz} + \sqrt[3]{xyz}$
Since $x, y, z \in \left[\frac{1}{3}, \frac{2}{3}\right]$,then $\frac{1}{3} \leq \sqrt[3]{xyz} \leq \frac{2}{3}$
Let $a = \sqrt[3]{xyz}$,then $a + \frac{2}{9a} \leq 1 \iff 9a^2 - 9a + 2 \leq 0 \iff 9\left(a - \frac{1}{3}\right)\left(a - \frac{2}{3}\right) \leq 0$,we're done since $\frac{1}{3} \leq a \leq \frac{2}{3}$

Problem 22 *'Endrit Fejzullahu'* (Endrit Fejzullahu): Let a, b, c be side lengths of a triangle,and β is the angle between a and c.Prove that

$$\frac{b^2 + c^2}{a^2} > \frac{2\sqrt{3}c\sin\beta - a}{b + c}$$

First Solution (Endrit Fejzullahu): According to the Weitzenbock's inequality we have
$a^2 + b^2 + c^2 \geq 4\sqrt{3}S$ and $S = \frac{ac\sin\beta}{2}$
Then
$b^2 + c^2 \geq 2\sqrt{3}ac\sin\beta - a^2$,dividing by a^2 , we have
$\frac{b^2 + c^2}{a^2} \geq \frac{2\sqrt{3}c\sin\beta - a}{a}$
Since $a < b + c \implies \frac{b^2 + c^2}{a^2} \geq \frac{2\sqrt{3}c\sin\beta - a}{a} > \frac{2\sqrt{3}c\sin\beta - a}{b + c}$

Problem 23 *'Dinu Serbanescu, Junior TST 2002, Romania'* (Hassan Al-Sibyani): If $a, b, c \in (0,1)$ Prove that:

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1$$

First Solution (FantasyLover):
Since $a, b, c \in (0,1)$, let us have $a = \sin^2 A, b = \sin^2 B, c = \sin^2 C$ where $A, B, C \in \left(0, \frac{\pi}{2}\right)$.
Then, we are to prove that $\sin A \sin B \sin C + \cos A \cos B \cos C < 1$.
Now noting that $\sin C, \cos C < 1$, we have $\sin A \sin B \sin C + \cos A \cos B \cos C < \sin A \sin B + \cos A \cos B = \cos(A - B) \le 1$. ■

Second Solution (Popa Alexandru): Cauchy-Schwartz and AM-GM works fine :

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} = \sqrt{a}\sqrt{bc} + \sqrt{1-a}\sqrt{(1-b)(1-c)} \le$$
$$\sqrt{a + (1-a)}\sqrt{bc + (1-b)(1-c)} = \sqrt{bc + (1-b)(1-c)} < 1$$

Third Solution (Redwane Khyaoui):

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} \le \frac{a + bc}{2} + \frac{1 - a + (1-b)(1-c)}{2}$$

So we only need to prove that $\frac{a + bc}{2} + \frac{1 - a + (1-b)(1-c)}{2} \le 1$
$\Leftrightarrow \frac{1}{b} + \frac{1}{c} \ge 2$ which is true since $a, b \in [0,1]$

Problem 24 '—' (FantasyLover): For all positive real numbers a, b, c, prove the following:

$$\frac{1}{\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}} - \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \ge \frac{1}{3}$$

First Solution (Popa Alexandru): Using p, q, r substitution ($p = a + b + c$, $q = ab + bc + ca$, $r = abc$) the inequality becomes :

$$\frac{3(p + q + r + 1)}{2p + q + r} \ge \frac{9 + 3r}{q} \Leftrightarrow pq + 2q^2 \ge 6pr + 9r$$

which is true because is well-known that $pq \ge 9r$ and $q^2 \ge 3pr$

Second Solution (Endrit Fejzullahu): After expanding the inequality is equivalent with :

$$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \ge \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right)$$

This is true by Chebyshev's inequality , so we're done .

Problem 25 *'Mihai Opincariu'* (Popa Alexandru): Let $a, b, c > 0$ such that $abc = 1$. Prove that :

$$\frac{ab}{a^2+b^2+\sqrt{c}} + \frac{bc}{b^2+c^2+\sqrt{a}} + \frac{ca}{c^2+a^2+\sqrt{b}} \leq 1$$

First Solution (FantasyLover):
We have $a^2+b^2+\sqrt{c} \geq 2ab+\sqrt{c} = \frac{2}{c}+\sqrt{c}$.
Hence, it suffices to prove that $\sum_{\text{cyc}} \frac{\frac{1}{a}}{\frac{2}{a}+\sqrt{a}} = \sum_{\text{cyc}} \frac{1}{2+a\sqrt{a}} \leq 1$.
Reducing to a common denominator, we prove that

$$\sum_{\text{cyc}} \frac{1}{2+a\sqrt{a}} = \frac{4(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})+ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca}+12}{4(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})+2(ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca})+9} \leq 1$$

.
Rearranging, it remains to prove that $ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca} \geq 3$.
Applying AM-GM, we have $ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca} \geq 3\sqrt[3]{a^2b^2c^2\sqrt{a^2b^2c^2}} = 3$, and we are done. ■

Second Solution (Popa Alexandru):

$$LHS \leq \sum_{cyc} \frac{ab}{2ab+\sqrt{c}} = \sum_{cyc} \frac{1}{2+c\sqrt{c}} = \sum_{cyc} \frac{1}{2+\frac{x}{y}} \leq RHS$$

Problem 26 *'Korea 2006 First Examination'* (FantasyLover): x, y, z are real numbers satisfying the condition $3x+2y+z=1$. Find the maximum value of

$$\frac{1}{1+|x|} + \frac{1}{1+|y|} + \frac{1}{1+|z|}$$

.

First Solution (dgreenb801):
We can assume x,y, and z are all positive, because if one was negative we could just make it positive, which would allow us to lessen the other two variables, making the whole sum larger.
Let $3x = a$, $2y = b$, $z = c$, then $a+b+c = 1$ and we have to maximize
$\frac{3}{a+3} + \frac{2}{b+2} + \frac{1}{c+1}$
Note that
$\left(\frac{3}{a+c+3}+1\right) - \left(\frac{3}{a+3}+\frac{1}{c+1}\right) = \frac{a^2c+ac^2+6ac+6c}{(a+3)(c+1)(a+c+3)} \geq 0$
So for fixed $a+c$, the sum is maximized when $c=0$.
We can apply the same reasoning to show the sum is maximized when $b=0$.
So the maximum occurs when $a=1$, $b=0$, $c=0$, and the sum is $\frac{11}{4}$.

Problem 27 *'Balkan Mathematical Olympiad 2006'* (dgreenb801):

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+a)}\geq\frac{3}{1+abc}$$

for all positive reals.

First Solution (Popa Alexandru):
AM-GM works :

$$(1+abc)\,LHS+3=\sum_{cyc}\frac{1+abc+a+ab}{a+ab}=\sum_{cyc}\frac{1+a}{ab+a}+\sum_{cyc}\frac{b(c+1)}{b+1}\geq\frac{3}{\sqrt[3]{abc}}+3\sqrt[3]{abc}\geq 6$$

Problem 28 *'Junior TST 2007, Romania'* (Popa Alexandru): Let $a,b,c>0$ such that $ab+bc+ca=3$. Show that :

$$\frac{1}{1+a^2(b+c)}+\frac{1}{1+b^2(c+a)}+\frac{1}{1+c^2(a+b)}\leq\frac{1}{abc}$$

First Solution (Endrit Fejzullahu): Since $ab+bc+ca=3\implies abc\leq 1$
Then $1+3a-abc\geq 3a$
Then
$\sum\frac{1}{1+3a-abc}\leq\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{ab+bc+ca}{3abc}=\frac{1}{abc}$

Problem 29 *'Lithuania 1987'* (Endrit Fejzullahu): Let a,b,c be positive real numbers .Prove that

$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\geq\frac{a+b+c}{3}$$

First Solution (dgreenb801): By Cauchy,
$\sum\frac{a^3}{a^2+ab+b^2}=\sum\frac{a^4}{a^3+a^2b+ab^2}\geq\frac{(a^2+b^2+c^2)^2}{a^3+b^3+c^3+a^2b+ab^2+b^2c+bc^2+c^2a+ca^2}$
This is $\geq\frac{a+b+c}{3}$ if
$(a^4+b^4+c^4)+2(a^2b^2+b^2c^2+c^2a^2)\geq(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)+(a^2bc+ab^2c+abc^2)$
This is equivalent to
$(a^2+b^2+c^2)(a^2+b^2+c^2-ab-bc-ca)\geq 0$
Which is true as $a^2+b^2+c^2-ab-bc-ca\geq 0\iff(a-b)^2+(b-c)^2+(c-a)^2\geq 0$

Second Solution (Popa Alexandru):
Since :

$$\frac{a^3-b^3}{a^2+ab+b^2}=a-b$$

and similars we get :

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} = \sum_{cyc} \frac{b^3}{a^2+ab+b^2} = \frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2}.$$

Now it remains to prove :

$$\frac{1}{2} \cdot \frac{a^3+b^3}{a^2+ab+b^2} \geq \frac{a+b}{6}$$

which is trivial .

Problem 30 *'—'* (dgreenb801): Given $ab+bc+ca=1$ Show that: $\frac{\sqrt{3a^2+b^2}}{ab} + \frac{\sqrt{3b^2+c^2}}{bc} + \frac{\sqrt{3c^2+a^2}}{ca} \geq 6\sqrt{3}$

First Solution (Hoang Quoc Viet):
Let's make use of Cauchy-Schwarz as demonstrated as follows

$$\frac{\sqrt{(3a^2+b^2)(3+1)}}{2ab} \geq \frac{3a+b}{2ab}$$

Thus, we have the following estimations

$$\sum_{cyc} \frac{\sqrt{3a^2+b^2}}{ab} \geq 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Finally, we got to prove that

$$\sum_{cyc} \frac{1}{a} \geq 3\sqrt{3}$$

However, from the given condition, we derive

$$abc \leq \frac{1}{3\sqrt{3}}$$

and

$$\sum_{cyc} \frac{1}{a} \geq 3\sqrt[3]{\frac{1}{abc}} \geq 3\sqrt{3}$$

Problem 31 *'Komal Magazine'* (Hoang Quoc Viet): Let a, b, c be real numbers. Prove that the following inequality holds

$$(a^2+2)(b^2+2)(c^2+2) \geq 3(a+b+c)^2$$

First Solution (Popa Alexandru): Cauchy-Schwartz gives :

$$(a^2+2)(b^2+2) = (a^2+1)(1+b^2)+a^2+b^2+3 \geq (a+b)^2 + \frac{1}{2}(a+b)^2 + 3 = \frac{3}{2}((a+b)^2+2)$$

And Cauchy-Schwartz again

$$(a^2+2)(b^2+2)(c^2+2) \geq \frac{3}{2}((a+b)^2+2)(2+c^2) \geq \frac{3}{2}(\sqrt{2}(a+b)+\sqrt{2}c)^2 = RHS$$

Problem 32 *'mateforum.ro'* (Popa Alexandru): Let $a, b, c \geq 0$ and $a + b + c = 1$. Prove that :

$$\frac{a}{\sqrt{b^2+3c}} + \frac{b}{\sqrt{c^2+3a}} + \frac{c}{\sqrt{a^2+3b}} \geq \frac{1}{\sqrt{1+3abc}}$$

First Solution (Endrit Fejzullahu):
Using Holder's inequality
$\left(\sum_{cyc} \frac{a}{\sqrt{b^2+3c}}\right)^2 \cdot \sum_{cyc} a(b^2+3c) \geq (a+b+c)^3 = 1$
It is enough to prove that
$1 + 3abc \geq \sum_{cyc} a(b^2+3c)$
Homogenise $(a+b+c)^3 = 1$,
Also after Homogenising $\sum_{cyc} a(b^2+3c) = a^2b + b^2c + c^2a + 9abc + 3\sum_{sym} a^2b$
$(a+b+c)^3 = a^3 + b^3 + c^3 + 6abc + 3\sum_{sym} a^2b$
It is enough to prove that
$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$
By AM-GM
$a^3 + a^3 + b^3 \geq 3a^2b$
$b^3 + b^3 + c^3 \geq 3b^2c$
$c^3 + c^3 + a^3 \geq 3c^2a$
Then $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$,done

Second Solution (manlio):
by holder.
$(LHS)^2(\sum a(b^2+ac) \geq (\sum a)^3$
so it suffices to prove
$\frac{1}{\sum a(b^2+3c)} \geq \frac{1}{1+3abc}$
that is
$\sum a^3 \geq \sum ab^2$ true by AM-GM

Third Solution (Apartim De):
$f(t) = \frac{1}{\sqrt{t}}; f'(t) < 0; f''(t) > 0$
Using Jensen with weights $a, b, c,$ we have
$af(b^2+3c) + bf(c^2+3a) + cf(a^2+3b) \geq f(ab^2+bc^2+ca^2+3ab+3bc+3ca)$
Now,
By Holder, $(a^3+b^3+c^3) = \sqrt[3]{(a^3+b^3+c^3)(b^3+c^3+a^3)(b^3+c^3+a^3)} \geq ab^2+bc^2+ca^2$
Again,
$3(a+b)(b+c)(c+a) + 3abc = 9abc + 3\sum_{sym} a^2b = 3(a+b+c)(ab+bc+ca) = 3(ab+bc+ca)$
$\therefore 1 + 3abc = (a+b+c)^3 + 3abc > ab^2 + bc^2 + ca^2 + 3ab + 3bc + 3ca$
$\therefore f(ab^2+bc^2+ca^2+3ab+3bc+3ca) > f(1+3abc)$
QED

Problem 33 *'Apartim De'* (Apartim De): If a, b, c, d be positive reals then prove that:

$$\frac{a^2+b^2}{ab+b^2}+\frac{b^2+c^2}{bc+c^2}+\frac{c^2+d^2}{cd+d^2}\geq\sqrt[3]{\frac{54a}{(a+d)}}$$

First Solution (Agr 94 Math):
Write the LHS as $\dfrac{(\frac{a}{b})^2+1}{(\frac{a}{b})+1}$+ two other similar termsfeel lazy to write them down. This is a beautiful application of Jensen's as the function for positve real t such that $f(t)=\dfrac{t^2+1}{t+1}$ is convex since $\left(\dfrac{t-1}{t+1}\right)^2\geq 0$.

Thus, we get that $LHS\geq\dfrac{\left(\frac{(\sum\frac{a}{b}-\frac{d}{a}}{3}\right)^2+1}{\sum\frac{a}{b}-\frac{d}{a}+3}$

I would like to write $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{d}=K$ for my convenience with latexing.

so we have $\dfrac{\left((\frac{K}{3})^2+1\right)}{K+3}=\dfrac{K+\frac{9}{K}}{1+\frac{3}{K}}\geq\dfrac{6}{1+\frac{3}{K}}$

This is from $K+\dfrac{9}{K}\geq 6$ by AM GM.

Now $K=\sum\dfrac{a}{b}-\dfrac{d}{a}\geq 3\left(\dfrac{d}{a}\right)^{\frac{1}{3}}$ by AM GM.

Thus, we have $\dfrac{6}{1+\frac{3}{K}}\geq\dfrac{6a^{\frac{1}{3}}}{a^{\frac{1}{3}}+d^{\frac{1}{3}}}\geq\dfrac{3a^{\frac{1}{3}}}{\frac{a+d}{2}^{\frac{1}{3}}}=\left(\dfrac{54a}{a+d}\right)^{\frac{1}{3}}$

Problem 34 '—' (Raghav Grover): If a and b are non negative real numbers such that $a\geq b$. Prove that

$$a+\frac{1}{b(a-b)}\geq 3$$

First Solution (Dimitris X):
If $a\geq 3$ the problem is obviously true.
Now for $a<3$ we have :
$a^2b-ab^2-3ab+3b^2+1\geq 0\iff ba^2-(b^2+3b)a+3b^2+1\geq 0$
It suffices to prove that $D\leq 0\iff b^4+6b^3+9b^2-12b^3-4b\iff b(b-1)^2(b-4)\leq 0$ which is true.

Second Solution (dgreenb801):
$a+\dfrac{1}{b(a-b)}=b+(a-b)+\dfrac{1}{b(a-b)}\geq 3$ by AM-GM

Third Solution (geniusbliss):
since $a\geq b$ we have $\frac{a}{2}\geq\sqrt{b(a-b)}$ square this and substitute in this denominator we get,
$\dfrac{a}{2}+\dfrac{a}{2}+\dfrac{4}{a^2}\geq 3$ by AM-GM so done.

Problem 35 *'Vasile Cirtoaje'* (Dimitris X):

$$(a^2-bc)\sqrt{b+c}+(b^2-ca)\sqrt{c+a}+(c^2-ab)\sqrt{a+b}\geq 0$$

First Solution (Popa Alexandru): Denote $\frac{a+b}{2}=x^2, \ldots$, then the inequality becomes :

$$\sum_{cyc} xy(x^3+y^3)\geq \sum_{cyc} x^2y^2(x+y)$$

which is equivalent with :

$$\sum_{cyc} xy(x+y)(x-y)^2\geq 0$$

Problem 36 *'Cezar Lupu'* (Popa Alexandru): Let $a, b, c > 0$ such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\sqrt{abc}$. Prove that:

$$abc\geq \sqrt{3(a+b+c)}$$

First Solution (Endrit Fejzullahu):
$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\sqrt{abc} \iff ab+bc+ca=abc\sqrt{abc}$
By Am-Gm
$(ab+bc+ca)^2\geq 3abc(a+b+c)$
Since ab + bc + ca = abc $\sqrt{abc}, wehave$
(abc) $^3 \geq 3abc(a+b+c) \implies abc\geq \sqrt{3(a+b+c)}$

Problem 37 *'Pham Kim Hung'* (Endrit Fejzullahu): Let a, b, c, d be positive real numbers satisfying $a+b+c+d=4$.Prove that

$$\frac{1}{11+a^2}+\frac{1}{11+b^2}+\frac{1}{11+c^2}+\frac{1}{11+d^2}\leq\frac{1}{3}$$

First Solution (Apartim De):
$f(x)=\dfrac{1}{11+x^2}\Leftrightarrow f''(x)=\dfrac{6(x^2-\frac{11}{3})}{(11+x^2)^3}$
$(x^2-\dfrac{11}{3})=\left(x-\sqrt{\dfrac{11}{3}}\right)\left(x+\sqrt{\dfrac{11}{3}}\right)$
If x $\in \left(-\sqrt{\dfrac{11}{3}},\sqrt{\dfrac{11}{3}}\right)$, $f''(x)<0$

Thus within the interval $\left(-\sqrt{\frac{11}{3}}, \sqrt{\frac{11}{3}}\right)$, the quadratic polynomial is negative

thereby making $f''(x) < 0,$ and thus $f(x)$ is concave within $\left(-\sqrt{\frac{11}{3}}, \sqrt{\frac{11}{3}}\right)$.

Let $a \leq b \leq c \leq d$. If all of $a, b, c, d \in \left(0, \sqrt{\frac{11}{3}}\right)$,

Then by Jensen, $f(a) + f(b) + f(c) + f(d) \leq 4f\left(\frac{a+b+c+d}{4}\right) = 4f(1) = \frac{4}{12} = \frac{1}{3}$

$f'(x) = \frac{-2x}{(11+x^2)^3} < 0$ (for all positive x)

At most 2 of a, b, c, d(namely c& d) can be greater than $\sqrt{\frac{11}{3}}$

In that case,

$f(a)+f(b)+f(c)+f(d) < f(a-1)+f(b-1)+f(c-3)+f(d-3) < 4f\left(\frac{a+b+c+d-8}{4}\right) = 4f(-1) = \frac{4}{12} = \frac{1}{3}$

QED

Problem 38 *'Crux Mathematicorum'* (Apartim De): Let R, r, s be the circumradius, inradius, and semiperimeter, respectively, of an acute-angled triangle. Prove or disprove that

$$s^2 \geq 2R^2 + 8Rr + 3r^2$$

.

When does equality occur?

First Solution (Virgil Nicula):

Proof. I"ll use the remarkable linear-angled identities $\left\|\begin{array}{cc} a^2+b^2+c^2 = 2\cdot\left(p^2-r^2-4Rr\right) & (1) \\ 4S = \left(b^2+c^2-a^2\right)\cdot\tan A & (2) \\ \sin 2A + \sin 2B + \sin 2C = \frac{2S}{R^2} & (3) \\ \cos A + \cos B + \cos C = 1 + \frac{r}{R} & (4) \end{array}\right\|$. Therefore,

$\sum a^2 = \sum\left(b^2+c^2-a^2\right) \stackrel{(2)}{=} 4S\cdot\sum\frac{\cos A}{\sin A} = 8S\cdot\sum\frac{\cos^2 A}{\sin 2A} \stackrel{C.B.S.}{\geq} 8S\cdot\frac{\left(\sum\cos A\right)^2}{\sum\sin 2A} \stackrel{(3)\wedge(4)}{=} 8S\cdot\frac{\left(1+\frac{r}{R}\right)^2}{\frac{2S}{R^2}} =$
$= 4(R{+}r)^2 \Longrightarrow \boxed{a^2+b^2+c^2 \geq 4(R+r)^2} \stackrel{(1)}{\Longrightarrow} 2\cdot\left(p^2-r^2-4Rr\right) \geq 4(R{+}r)^2 \Longrightarrow \boxed{p^2 \geq 2R^2+8Rr+3r^2}$.

Problem 39 *'Russia 1978'* (Endrit Fejzullahu): Let $0 < a < b$ and $x_i \in [a, b]$.Prove that

$$(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \leq \frac{n^2(a+b)^2}{4ab}$$

First Solution (Popa Alexandru):
We will prove that if $a_1, a_2, \ldots, a_n \in [a, b](0 < a < b)$ then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \frac{(a+b)^2}{4ab}n^2$$

$$P = (a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) = \left(\frac{a_1}{c} + \frac{a_2}{c} + \cdots + \frac{a_n}{c}\right)\left(\frac{c}{a_1} + \frac{c}{a_2} + \cdots + \frac{c}{a_n}\right) \leq$$

$$\leq \frac{1}{4}\left(\frac{a_1}{c} + \frac{c}{a_1} + \frac{a_2}{c} + \frac{c}{a_2} + \cdots + \frac{a_n}{c} + \frac{c}{a_n}\right)^2$$

Function $f(t) = \frac{c}{t} + \frac{t}{c}$ have its maximum on $[a, b]$ in a or b. We will choose c such that $f(a) = f(b), c = \sqrt{ab}$.
Then $f(t) \leq \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}$. Then

$$P \leq n^2\left(\sqrt{\frac{a}{b}} + \sqrt{b\frac{b}{a}}\right)^2 \cdot \frac{1}{4} = n^2\frac{(a+b)^2}{4ab}$$

Problem 40 *'—'* (keyree10): a, b, c are non-negative reals. Prove that

$$81abc \cdot (a^2 + b^2 + c^2) \leq (a + b + c)^5$$

First Solution (Popa Alexandru):

$$81abc(a^2 + b^2 + c^2) \leq 27\frac{(ab + bc + ca)^2}{a + b + c}(a^2 + b^2 + c^2) \leq (a + b + c)^5$$

By p,q,r the last one is equivalent with :

$$p^6 - 27q^2p^2 + 54q^3 \geq 0 \Leftrightarrow (p^2 - 3q)^2 \geq 0$$

Problem 41 *'mateforum.ro'* (Popa Alexandru): Let $a, b, c > 0$ such that $a^3 + b^3 + 3c = 5$. Prove that :

$$\sqrt{\frac{a+b}{2c}} + \sqrt{\frac{b+c}{2a}} + \sqrt{\frac{c+a}{2b}} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

First Solution (Sayan Mukherjee):
$a^3 + 1 + 1 + b^3 + 1 + 1 + 3c = 9 \geq 3a + 3b + 3c \implies a + b + c \leq 3$(AM-GM)
Also $abc \leq \left(\frac{a+b+c}{3}\right) = 1$(AM-GM)

Applying CS on the LHS;
$$\left[\sum\sqrt{\frac{a+b}{2c}}\right]^2 \leq (a+b+c)\left(\sum\frac{1}{a}\right)$$
It is left to prove that $\sum\frac{1}{a}\geq\sum a$
But this $\implies \sum\frac{1}{a}\geq 3$
Which is perfextly true, as from AM-GM on the LHS;
$\sum\frac{1}{a}\geq 3\sqrt[3]{\frac{1}{abc}}\geq 3\,[\because abc\leq 1]$

Problem 42 '—' (Sayan Mukherjee): Let $a,b,c>0$ PT: If a,b,c satisfy $\sum\frac{1}{a^2+1}=\frac{1}{2}$ Then we always have:
$$\sum\frac{1}{a^3+2}\leq\frac{1}{3}$$

First Solution (Endrit Fejzullahu):
$$a^3+a^3+1\geq 3a^2 \implies a^3+a^3+1+3\geq 3a^2+3 \implies \frac{1}{3(a^2+1)}\geq\frac{1}{2(a^3+2)} \implies \frac{1}{3}\geq\sum\frac{1}{a^3+2}$$

Problem 43 *'Russia 2002'* (Endrit Fejzullahu): Let a,b,c be positive real numbers with sum 3.Prove that
$$\sqrt{a}+\sqrt{b}+\sqrt{c}\geq ab+bc+ca$$

First Solution (Sayan Mukherjee):
$2(ab+bc+ca)=9-(a^2+b^2+c^2)$
as $a+b+c=9$. Hence the inequality
$\implies 2\sum\sqrt{a}\geq 9-(a^2+b^2+c^2)$
$\implies \sum(a^2+\sqrt{a}+\sqrt{a})\geq 9$
Perfectly true from AGM as :
$a^2+\sqrt{a}+\sqrt{a}\geq 3a$

Problem 44 *'India 2002'* (Raghav Grover): For any natural number n prove that
$$\frac{1}{2}\leq\frac{1}{n^2+1}+\frac{1}{n^2+2}+....+\frac{1}{n^2+n}\leq\frac{1}{2}+\frac{1}{2n}$$

First Solution (Sayan Mukherjee):
$\sum_{k=1}^{n} \frac{k}{n^2+k} = \sum_{k=1}^{n} \frac{k^2}{n^2k+k^2} \geq \frac{\sum_{k=1}^{n} k}{n^2 \sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2} = \frac{3(n^2+n)}{2(3n^2+2n+1)} > \frac{1}{2}$
As it is equivalent to :$3n^2+3n > 3n^2+2n+1 \implies n > 1$
For the 2nd part;
$\sum \frac{k}{n^2+k} = \sum 1 - \frac{n^2}{n^2+k} = n - n^2 \sum \frac{1}{n^2+k}$ and then use AM-HM for $\sum \frac{1}{n^2+k}$ So we get the desired result.

Problem 45 *'—'* (Sayan Mukherjee): For $a,b,c>0; a^2+b^2+c^2=1$ find P_{min} if:

$$P = \sum_{cyc} \frac{a^2b^2}{c^2}$$

First Solution (Endrit Fejzullahu):
Let $x = \frac{ab}{c}, y = \frac{bc}{a} and z = \frac{ca}{b}$
Then Obviously By Am-Gm
$x^2+y^2+z^2 \geq xy+yz+zx = a^2+b^2+c^2 = 1$,then $P_{min} = 1$

Problem 46 *'—'* (Endrit Fejzullahu): Let a,b,c be positive real numbers such that $a+b+c=3$ .Prove that

$$\frac{a^2}{a+2b^2} + \frac{b^2}{b+2c^2} + \frac{c^2}{c+2a^2} \geq 1$$

First Solution (Popa Alexandru): We start with a nice use of AM-GM :

$$\frac{a^2}{a+2b^2} = \frac{a^2+2ab^2-2ab^2}{a+2b^2} = \frac{a(a+2b^2)}{a+2b^2} - \frac{2ab^2}{a+2b^2} \geq a - \frac{2}{3}\sqrt[3]{a^2b^2}$$

Suming the similars we need to prove :

$$\sqrt[3]{a^2b^2} + \sqrt[3]{b^2c^2} + \sqrt[3]{c^2a^2} \leq 3$$

By AM-GM :

$$\sum_{cyc} \sqrt[3]{a^2b^2} \leq \sum_{cyc} \frac{2ab+1}{3} = \frac{1}{3}\left(2\sum_{cyc} ab + 3\right) \leq 3 \Leftrightarrow ab+bc+ca \leq 3 \Leftrightarrow 3(ab+bc+ca) \leq (a+b+c)^2$$

Problem 47 *'mateforum.ro'* (Popa Alexandru): Let $a,b,c>0$ such that $a+b+c \leq \frac{3}{2}$. Prove that :

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2}$$

First Solution:
WLOG $a \geq b \geq c$
Then By Chebyshev's inequality we have
$RHS \geq \frac{1}{3} \cdot LHS \cdot \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right)$
It is enough to show that
$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq 3$
By Cauchy Schwartz
$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{9}{2(a+b+c)} \geq 3 \iff a+b+c \leq \frac{3}{2}$, done

Problem 48 *'USAMO 2003'* (Hassan Al-Sibyani): Let a, b, c be positive real numbers. Prove that:

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+a+c)^2}{2b^2+(a+c)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8$$

First Solution (Popa Alexandru):
Suppose $a+b+c=3$
The inequality is equivalent with :

$$\frac{(a+3)^2}{2a^2+(3-a)^2} + \frac{(b+3)^2}{2b^2+(3-b)^2} + \frac{(c+3)^2}{2c^2+(3-c)^2} \leq 8$$

For this we prove :

$$\frac{(a+3)^2}{2a^2+(3-a)^2} \leq \frac{4}{3}a + \frac{4}{3} \Leftrightarrow 3(4a+3)(a-1)^2 \geq 0$$

done.

Problem 49 *'Marius Mainea'* (Popa Alexandru): Let $x, y, z > 0$ such that $x+y+z = xyz$. Prove that :

$$\frac{x+y}{1+z^2} + \frac{y+z}{1+x^2} + \frac{z+x}{1+y^2} \geq \frac{27}{2xyz}$$

First Solution (socrates): Using Cauchy-Schwarz and AM-GM inequality we have:

$$\frac{x+y}{1+z^2}+\frac{y+z}{1+x^2}+\frac{z+x}{1+y^2} = \frac{(x+y)^2}{x+y+(x+y)z^2}+\frac{(y+z)^2}{y+z+(y+z)x^2}+\frac{(z+x)^2}{z+x+(z+x)y^2} \geq \frac{((x+y)+(y+z)+(z+x))^2}{2(x+y+z)+\sum x^2(y+z)} =$$

$$\frac{4(x+y+z)^2}{2xyz+\sum x^2(y+z)} = \frac{4(x+y+z)^2}{\prod(x+y)} \geq 4(x+y+z)^2 \cdot \frac{27}{8} \frac{1}{(x+y+z)^3} = \frac{27}{2xyz}$$

Problem 50 '—' (socrates): Let $a, b, c > 0$ such that $ab + bc + ca = 1$. Prove that :

$$abc(a+\sqrt{a^2+1})(b+\sqrt{b^2+1})(c+\sqrt{c^2+1}) \le 1$$

First Solution (dgreenb801): Note that $\sqrt{a^2+1} = \sqrt{a^2+ab+bc+ca} = \sqrt{(a+b)(a+c)}$
Also, by Cauchy, $(a+\sqrt{(a+b)(a+c)})^2 \le (a+(a+b))(a+(a+c)) = (2a+b)(2a+c)$
So after squaring both sides of the inequality, we have to show

$$1 = (ab+bc+ca)^6 \ge a^2b^2c^2(2a+b)(2a+c)(2b+a)(2b+c)(2c+a)(2c+b) =$$
$$(2ac+bc)(2ab+bc)(2bc+ac)(2ab+ac)(2cb+ab)(2ca+ab)$$

, which is true by AM-GM.

Problem 51 *'Asian Pacific Mathematics Olympiad'* (dgreenb801): Let $a, b, c > 0$ such that $abc = 8$. Prove that:

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \ge \frac{4}{3}$$

First Solution (Popa Alexandru): By AM-GM :

$$\sqrt{a^3+1} \le \frac{a^2+2}{2}.$$

Then we have :

$$\sum_{cyc} \frac{a^2}{\sqrt{(a^3+1)(b^3+1)}} \ge 4\sum_{cyc} \frac{a^2}{(a^2+2)(b^2+2)}$$

So we need to prove

$$\sum_{cyc} \frac{a^2}{(a^2+2)(b^2+2)} \ge \frac{1}{3}.$$

which is equivalent with :

$$a^2b^2 + b^2c^2 + c^2a^2 + 2(a^2+b^2+c^2) \ge 72.$$

, true by AM-GM

Problem 52 *'Lucian Petrescu'* (Popa Alexandru): Prove that in any acute-angled triangle ABC we have :

$$\frac{a+b}{\cos C} + \frac{b+c}{\cos A} + \frac{c+a}{\cos B} \ge 4(a+b+c)$$

First Solution (socrates): The inequality can be rewritten as

$$\frac{ab(a+b)}{a^2+b^2-c^2} + \frac{bc(b+c)}{b^2+c^2-a^2} + \frac{ca(c+a)}{c^2+a^2-b^2} \ge 2(a+b+c)$$

or

$$\sum_{cyclic} a(\frac{b^2}{a^2+b^2-c^2}+\frac{c^2}{c^2+a^2-b^2}) \geq 2(a+b+c)$$

Applying Cauchy Schwarz inequality we get

$$\sum_{cyclic} a(\frac{b^2}{a^2+b^2-c^2}+\frac{c^2}{c^2+a^2-b^2}) \geq \sum a\frac{(b+c)^2}{(a^2+b^2-c^2)+(c^2+a^2-b^2)}$$

or

$$\sum_{cyclic} a(\frac{b^2}{a^2+b^2-c^2}+\frac{c^2}{c^2+a^2-b^2}) \geq \sum \frac{(b+c)^2}{2a}$$

So, it is enough to prove that

$$\sum \frac{(b+c)^2}{2a} \geq 2(a+b+c)$$

which is just CS as above.

Second Solution (Mateescu Constantin):
From the law of cosinus we have $a = b\cos C + c\cos B$and $c = b\cos A + a\cos B$.
$\Longrightarrow \dfrac{a+c}{\cos B} = a+c+\dfrac{b(\cos A+\cos C)}{\cos B} = a+c+2R\tan B(\cos A+\cos C)$. Then:
$\sum \dfrac{a+c}{\cos B} = 2(a+b+c)+2R\sum \tan B\cdot(\cos A+\cos C)$ and the inequality becomes:
$2R\sum \tan B(\cos A+\cos C) \geq 2\sum a = 4R\sum \sin A$
$\Longleftrightarrow \sum \tan B\cos A + \sum \tan B\cos C \geq 2\sum \sin A = 2\sum \tan A\cos A$
Wlog assume that $A \leq B \leq C$. Then $\cos A \geq \cos B \geq \cos C$ and $\tan A \leq \tan B \leq \tan C$, so
$\sum \tan A\cos A \leq \sum \tan B\cos A$ and $\sum \tan A\cos A \leq \sum \tan B\cos C$, according to rearrangement inequality
.
Adding up these 2 inequalities yields the conclusion.

Third Solution (Popa Alexandru):

$$LHS \geq \frac{(2(a+b+c))^2}{(a+b)\cos C+(b+c)\cos A+(c+a)\cos B} = 4(a+b+c)$$

Problem 53 *'—'* (socrates): Given $x_1, x_2, .., x_n > 0$ such that $\sum_{i=1}^{n} x_i = 1$, prove that

$$\sum_{i=1}^{n} \frac{x_i+n}{1+x_i^2} \leq n^2$$

First Solution (Hoang Quoc Viet):
Without loss of generality, we may assume that

$$x_1 \geq x_2 \geq \cdots \geq x_n$$

Therefore, we have

$$nx_1 - 1 \geq nx_2 - 1 \geq \cdots \geq nx_n - 1$$

and

$$\frac{x_1}{x_1^2+1} \geq \frac{x_2}{x_2^2+1} \geq \cdots \geq \frac{x_n}{x_n^2+1}$$

Hence, by Chebyshev inequality, we get

$$\sum_{i=1}^{n} \frac{(nx_i - 1)x_i}{x_i^2+1} \geq \frac{1}{n}\left[n\left(\sum_{i=1}^{n} x_i\right) - n\right]\left(\sum_{i=1}^{n} \frac{x_i}{x_i^2+1}\right) = 0$$

Thus, we get the desired result, which is

$$\sum_{i=1}^{n} \frac{x_i + n}{1 + x_i^2} \leq n^2$$

Problem 54 *'Hoang Quoc Viet'* (Hoang Quoc Viet): Let a, b, c be positive reals satisfying $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a^3}{2b^2+c^2} + \frac{b^3}{2c^2+a^2} + \frac{c^3}{2a^2+b^2} \geq 1$$

First Solution (Hoang Quoc Viet):
Using Cauchy Schwartz, we get

$$\sum_{cyc} \frac{a^3}{2b^2+c^2} \geq \frac{(a^2+b^2+c^2)^2}{2\sum_{cyc} ab^2 + \sum_{cyc} ac^2}$$

Hence, it suffices to prove that

$$ab^2 + bc^2 + ca^2 \leq 3$$

and

$$a^2b + b^2c + c^2a \leq 3$$

However, applying Cauchy Schwartz again, we obtain

$$a(ab) + b(bc) + c(ca) \leq \sqrt{(a^2+b^2+c^2)\left((ab)^2 + (bc)^2 + (ca)^2\right)}$$

In addition to that, we have

$$(ab)^2 + (bc)^2 + (ca)^2 \leq \frac{(a^2+b^2+c^2)^2}{3}$$

Hence, we complete our proof.

Problem 55 *'Iran 1998'* (saif): Let $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ prove that:

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{x-1} + \sqrt{z-1}$$

First Solution (beautifulliar):
Note that you can substitute $\sqrt{x-1}=a, \sqrt{y-1}=b, \sqrt{z-1}=c$ then you need to prove that $\sqrt{a^2+b^2+c^2+3} \geq a+b+c$ while you have $\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}=2$ or equivalently $a^2b^2+b^2c^2+c^2a^2+2a^2b^2c^2=1$. next, substitute $ab=\cos x, bc=\cos y, ca=\cos z$ where x,y,z are angles of triangle. since you need to prove that $\sqrt{a^2+b^2+c^2+3} \geq a+b+c$ then you only need to prove that $3 \geq 2(ab+bc+ca)$ or $\cos x+\cos B+\cos C \leq \frac{3}{2}$ which is trivial.

Second Solution (Sayan Mukherjee):
$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2 \implies \sum \frac{x-1}{x}=1$
Hence $(x+y+z)\sum \frac{x-1}{x} \geq \left(\sum \sqrt{x-1}\right)^2$(From CS)
$\implies \sqrt{x+y+z} \geq \sum \sqrt{x-1}$

Problem 56 *'IMO 1998'* (saif): Let $a_1,a_2,...,a_n>0$ such that $a_1+a_2+...+a_n<1$. prove that

$$\frac{a_1.a_2...a_n(1-a_1-a_2-...-a_n)}{(a_1+a_2+...+a_n)(1-a_1)(1-a_2)...(1-a_n)} \leq \frac{1}{n^{n-1}}$$

First Solution (beautifulliar):
Let $a_{n+1}=1-a_1-a_2-\cdots-a_n$ the we arrive at $\frac{a_1a_2\ldots a_na_{n+1}}{(1-a_1)(1-a_2)\ldots(1-a_n)} \leq \frac{1}{n^{n+1}}$(it should be n^{n+1} right?) which follows from am-gm $a_1+a_2+\cdots+a_n \geq n\sqrt[n]{a_1a_2\ldots a_n}, a_2+a_3+\cdots+a_{n+1} \geq n\sqrt[n]{a_2a_3\ldots a_{n+1}}, a_1+a_3+\cdots+a_{n+1} \geq n\sqrt[n]{a_1a_3\ldots a_{n+1}}$, and so on... you will find it easy.

Problem 57 *'—'* (beautifulliar): Let n be a positive integer. If $x_1,x_2,\ldots,x_n$ are real numbers such that $x_1+x_2+\cdots+x_n=0$ and also $y=\max\{x_1,x_2,\ldots,x_n\}$ and also $z=\min\{x_1,x_2,\ldots,x_n\}$, prove that

$$x_1^2+x_2^2+\cdots+x_n^2+nyz \leq 0$$

First Solution (socrates):
$(x_i-y)(x_i-z) \leq 0\ \forall i=1,2,...,n$ so $\sum_{i=1}^{n}(x_i^2+yz) \leq \sum_{i=1}^{n}(y+z)x_i=0$ and the conclusion follows.

Problem 58 *'Pham Kim Hung'* (Endrit Fejzullahu): Suppose that x,y,z are positive real numbers and $x^5+y^5+z^5=3$.Prove that

$$\frac{x^4}{y^3}+\frac{y^4}{z^3}+\frac{z^4}{x^3} \geq 3$$

First Solution (Popa Alexandru):
By a nice use of AM-GM we have :

$$10\left(\frac{x^4}{y^3}+\frac{y^4}{z^3}+\frac{z^4}{x^3}+3(x^5+y^5+z^5)^2 \geq 19(\sqrt[19]{x^{100}}+\sqrt[19]{y^{100}}+\sqrt[19]{z^{100}})\right.$$

So it remains to prove :

$$3+19(\sqrt[19]{x^{100}}+\sqrt[19]{y^{100}}+\sqrt[19]{z^{100}}) \geq 20(x^5+y^5+z^5)$$

which is true by AM-GM .

Problem 59 *'China 2003'* (bokagadha): x, y, and z are positive real numbers such that $x+y+z=xyz$. Find the minimum value of:

$$x^7(yz-1)+y^7(xz-1)+z^7(xy-1)$$

First Solution (Popa Alexandru):
By AM-GM and the condition you get

$$xyz \geq 3\sqrt{3}$$

Also observe that the condition is equivalent with

$$yz-1=\frac{y+z}{x}$$

So the

$$LHS=x^6(y+z)+y^6(z+x)+z^6(x+y) \geq 6\sqrt[6]{x^{14}y^{14}z^{14}} \geq 216\sqrt{3}$$

Problem 60 *'Austria 1990'* (Rofler):

$$\sqrt{2\sqrt{3\sqrt{4\sqrt{...\sqrt{N}}}}}<3 --- \forall N \in \mathbb{N}^{\geq 2}$$

First Solution (Brut3Forc3): We prove the generalization $\sqrt{m\sqrt{(m+1)\sqrt{\ldots\sqrt{N}}}} < m+1$, for $m+2$. For $m=N$, this is equivalent to $\sqrt{N} < N+1$, which is clearly true. We now induct from $m=N$ down. Assume that $\sqrt{(k+1)\sqrt{(k+2)\ldots\sqrt{N}}} < k+2$. Multiplying by k and taking the square root gives $\sqrt{k\sqrt{(k+1)\sqrt{\ldots\sqrt{N}}}} < \sqrt{k(k+2)} < k+1$, completing the induction.

Problem 61 *'Tran Quoc Anh'* (Sayan Mukherjee): Given $a, b, c \geq 0$ Prove that:

$$\sum_{cyc}\sqrt{\frac{a+b}{b^2+4bc+c^2}} \geq \frac{3}{\sqrt{a+b+c}}$$

First Solution (Hoang Quoc Viet): Using Cauchy inequality, we get

$$\sum_{cyc}\sqrt{\frac{a+b}{b^2+4bc+c^2}} \ge \sum_{cyc}\sqrt{\frac{2(a+b)}{3(b+c)^2}} \ge \sum_{cyc} 3\sqrt[6]{\frac{8}{27(a+b)(b+c)(c+a)}}$$

However, we have

$$(a+b)(b+c)(c+a) \le \frac{8(a+b+c)^3}{27}$$

Hence, we complete our proof here.

Problem 62 *'Popa Alexandru'* (Popa Alexandru): Let $a, b, c > 0$ such that $a + b + c = 1$. Show that :

$$\frac{a^2+ab}{1-a^2}+\frac{b^2+bc}{1-b^2}+\frac{c^2+ca}{1-c^2} \ge \frac{3}{4}$$

First Solution (Endrit Fejzullahu): Let $a+b = x$, $b+c = y$, $c+a = z$,given inequality becomes

$$\sum \frac{x}{y}+\sum \frac{x}{x+y} \ge \frac{9}{2}$$

Then By Cauchy Schwartz

$$LHS = \sum_{cyc}\frac{x}{y}+\sum_{cyc}\frac{x}{x+y} \ge \frac{(\sum x)^4}{(\sum xy)(\sum xy+\sum x^2)} \ge \frac{8(\sum x)^4}{(\sum xy+(\sum x)^2)^2} \ge \frac{9}{2}$$

Second Solution (Hoang Quoc Viet): Without too many technical terms, we have

$$\left(\sum_{cyc}\frac{x+y}{4y}+\sum_{cyc}\frac{x}{x+y}\right) \ge \sum_{cyc}\sqrt{\frac{x}{y}} \ge 3$$

Therefore, it is sufficient to check that

$$\sum_{cyc}\frac{x}{y} \ge 3$$

which is Cauchy inequality for 3 positive reals.

Problem 63 *'India 2007'* (Sayan Mukherjee): For positive reals a, b, c. Prove that:

$$(a+b+c)^2(ab+bc+ca)^2 \le 3(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ac+a^2)$$

First Solution (Endrit Fejzullahu): I've got an SOS representation of :

$$RHS-LHS = \frac{1}{2}((x+y+z)^2(x^2y^2+y^2z^2+z^2x^2-xyz(x+y+z))+(xy+yz+zx)(x^2+y^2+z^2-xy-yz-zx))$$

So I may assume that the inequality is true for reals

Problem 64 *'Popa Alexandru'* (Endrit Fejzullahu): Let $a, b, c > 0$ such that $(a+b)(b+c)(c+a) = 1$. Show that :

$$\frac{3}{16abc} \geq a+b+c \geq \frac{2}{3} \geq \frac{16abc}{3}$$

First Solution (Apartim De): By AM-GM, $\frac{(a+b)(b+c)(c+a)}{8} \geq abc \Leftrightarrow \frac{1}{8} \geq abc \Leftrightarrow \frac{2}{3} \geq \frac{16}{3}abc$ By AM-GM,
$(a+b)+(b+c)+(c+a) \geq 3$
$\Leftrightarrow (a+b+c) \geq \frac{3}{2} > \frac{2}{3}$
Lemma:We have for any positive reals x, y, z and vectors $\overrightarrow{MA}, \overrightarrow{MB}, \overrightarrow{MC}$
$\left(x\overrightarrow{MA} + y\overrightarrow{MB} + z\overrightarrow{MC}\right)^2 \geq 0$
$\Leftrightarrow (x+y+z)(xMA^2 + yMB^2 + zMC^2) \geq \left(xyAB^2 + yzBC^2 + zxCA^2\right)$
Now taking $x = y = z = 1$ and M to be the circumcenter of the triangle with sides p, q, r such that $pqr = 1$, and the area of the triangle=Δ, we have by the above lemma,
$9R^2 \geq p^2 + q^2 + r^2 \geq 3\,(pqr)^{\frac{2}{3}} = 3 \Leftrightarrow 3R^2 \geq 1 \Leftrightarrow 16\mathbf{\Delta^2} \leq 3$
$\Leftrightarrow (p+q+r)(p+q-r)(q+r-p)(r+p-q) \leq 3$
Now plugging in the famous Ravi substitution i.e,
$p = (a+b); q = (b+c); r = (c+a)$
$\Leftrightarrow (a+b+c) \leq \frac{3}{16abc}$

Problem 65 *'IMO 1988 shortlist'* (Apartim De): In the plane of the acute angled triangle ΔABC, L is a line such that u, v, w are the lengths of the perpendiculars from A, B, C respectively to L. Prove that

$$u^2 \tan A + v^2 \tan B + w^2 \tan C \geq 2\Delta$$

where Δ is the area of the triangle.

First Solution (Hassan Al-Sibyani): Consider a Cartesian system with the x-axis on the line BC and origin at the foot of the perpendicular from A to BC, so that A lies on the y-axis. Let A be $(0, \alpha), B(-\beta, 0), C(\gamma, 0)$, where $\alpha, \beta, \gamma > 0$ (because ABC is acute-angled). Then
$\tan B = \frac{\alpha}{\beta}$
$\tan C = \frac{\alpha}{\gamma}$
$\tan A = -\tan(B+C) = \frac{\alpha(\beta+\gamma)}{\alpha^2 - \beta\gamma}$
here $\tan A > 0$,so $\alpha^2 > \beta\gamma$. Let L have equation $x\cos\theta + y\sin\theta + p = 0$
Then
$u^2 \tan A + v^2 \tan B + w^2 \tan C$
$= \frac{\alpha(\beta+\gamma)}{\alpha^2 - \beta\gamma}(\alpha\sin\theta + p)^2 + \frac{\alpha}{\beta}(-\beta\cos\theta + p)^2 + \frac{\alpha}{\gamma}(\gamma\cos\theta + p)^2$
$= \alpha^2\sin^2\theta + 2\alpha p\sin\theta + p^2)\frac{\alpha(\beta+\gamma)}{\alpha^2 - \beta\gamma} + \alpha(\beta+\gamma)\cos^2\theta + \frac{\alpha(\beta+\gamma)}{\beta\gamma}p^2$
$= \frac{\alpha(\beta+\gamma)}{\beta\gamma(\alpha^2 - \beta\gamma)}(\alpha^2p^2 + 2\alpha p\beta\gamma\sin\theta + \alpha^2\beta\gamma\sin^2\theta + \beta\gamma(\alpha^2 - \beta\gamma)\cos^2\theta)$

$$= \frac{\alpha(\beta+\gamma)}{\beta\gamma(\alpha^2-\beta\gamma)}[(\alpha p+\beta\gamma\sin\theta)^2+\beta\gamma(\alpha^2-\beta\gamma)] \geq \alpha(\beta+\gamma) = 2\Delta$$

with equality when $\alpha p+\beta\gamma\sin\theta = 0$, i.e., if and only if L passes through $(0, \beta\gamma/\alpha)$, which is the orthocenter of the triangle.

Problem 66 *'—'* (Hassan Al-Sibyani): For positive real number a, b, c such that $abc \leq 1$, Prove that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq a+b+c$$

First Solution (Endrit Fejzullahu): It is easy to prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b+c}{\sqrt[3]{abc}}$$

and since $abc \leq 1$ then $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq a+b+c$,as desired .

Problem 67 *'Endrit Fejzullahu'* (Endrit Fejzullahu): Let a, b, c, d be positive real numbers such that $a+b+c+d = 4$.Find the minimal value of :

$$\sum_{cyc} \frac{a^4}{(b+1)(c+1)(d+1)}$$

First Solution (Sayan Mukherjee): From AM-GM,

$$P = \sum_{cyc}\left[\frac{a^4}{(b+1)(c+1)(d+1)}+\frac{b+1}{16}+\frac{c+1}{16}+\frac{d+1}{16}\right] \geq \sum_{cyc}\left[\frac{4a}{8}\right]$$

$$\Longrightarrow \sum_{cyc}\left[\frac{a^4}{(b+1)(c+1)(d+1)}\right] \geq \frac{a+b+c+d}{2} - \frac{3}{16}(a+b+c+d) - 4\cdot\frac{3}{16}$$

$$\therefore \sum_{cyc}\left[\frac{a^4}{(b+1)(c+1)(d+1)}\right] \geq \frac{4}{2}-\frac{6}{4} = 2-\frac{3}{2} = \frac{1}{2}$$

So $P_{min} = \dfrac{1}{2}$

Problem 68 *'—'* (Sayan Mukherjee): Prove that

$$\sum_{cyc} \frac{x}{7+z^3+y^3} \leq \frac{1}{3} \forall 1 \geq x, y, z \geq 0$$

First Solution (Toang Huc Khein):
$\sum \frac{x}{7+y^3+z^3} \leq \sum \frac{x}{6+x^3+y^3+z^3} \leq \sum \frac{x}{3(x+y+z)} = \frac{1}{3}$ because $x^3+y^3+z^3+6 \geq 3(x+y+z) \Leftrightarrow$
$\sum (x-1)^2(x+2) \geq 0$

Problem 69 *'Marius Maine'* (Toang Huc Khein): Let $x, y, z > 0$ with $x+y+z=1$. Then :

$$\frac{x^2-yz}{x^2+x} + \frac{y^2-zx}{y^2+y} + \frac{z^2-xy}{z^2+z} \leq 0$$

First Solution (Endrit Fejzullahu):

$$\frac{x^2-yz}{x^2+x} = 1 - \frac{x+yz}{x^2+x}$$

Inequality is equivalent with :

$$\sum_{cyc} \frac{x+yz}{x^2+x} \geq 3$$

$$\sum_{cyc} \frac{x+yz}{x^2+x} = \sum_{cyc} \frac{1}{x+1} + \sum_{cyc} \frac{yz}{x^2+x}$$

By Cauchy-Schwarz inequality

$$\sum_{cyc} \frac{1}{x+1} \geq \frac{9}{4}$$

And

$$\sum_{cyc} \frac{yz}{x^2+x} \geq \frac{(xy+yz+zx)^2}{x^2yz+y^2xz+z^2xy+3xyz} = \frac{(xy+yz+zx)^2}{4xyz} \geq \frac{3}{4} \iff (xy+yz+zx)^2 \geq 3xyz$$

This is true since $x+y+z=1$ and $(xy+yz+zx)^2 \geq 3xyz(x+y+z)$

Second Solution (Endrit Fejzullahu): Let us observe that by Cauchy-Schwartz we have :

$$\sum_{cyc} \frac{x+yz}{x^2+x} = \sum_{cyc} \frac{x(x+y+z)+yz}{x(x+1)} = \sum_{cyc} \frac{(x+y)(x+z)}{x(x+y+x+z)} = \sum_{cyc} \frac{1}{\frac{x}{x+y}+\frac{x}{x+z}} \geq \frac{(1+1+1)^2}{3} = \frac{9}{3} = 3$$

Then we can conclude that :

$$LHS = \sum_{cyc} \frac{x^2+x}{x^2+x} - \sum_{cyc} \frac{x+yz}{x^2+x} \leq 3-3 = 0$$

Problem 70 *'Claudiu Mandrila'* (Endrit Fejzullahu): Let $a, b, c > 0$ such that $abc = 1$. Prove that :

$$\frac{a^{10}}{b+c} + \frac{b^{10}}{c+a} + \frac{c^{10}}{a+b} \geq \frac{a^7}{b^7+c^7} + \frac{b^7}{c^7+a^7} + \frac{c^7}{a^7+b^7}$$

First Solution (Sayan Mukherjee): Since $abc = 1$ so, we have:

$$\sum \frac{a^{10}}{b+c} = \sum \frac{a^7}{b^3c^3(b+c)} \geq \sum \frac{a^7}{\frac{1}{64}(b+c)^6 \cdot (b+c)}$$

So we are only required to prove that: $(b+c)^7 \leq 2^6b^7 + 2^6c^7$
But, from Holder;
$(b^7+c^7)^{\frac{1}{7}}(1+1)^{\frac{6}{7}} \geq b+c$ Hence we are done

Problem 71 *'Hojoo Lee, Crux Mathematicorum'* (Sayan Mukherjee): Let $a, b, c \in \mathbb{R}^+$;Prove that:

$$\frac{2}{abc}(a^3+b^3+c^3) + \frac{9(a+b+c)^2}{a^2+b^2+c^2} \geq 33$$

First Solution (Endrit Fejzullahu): Inequality is equivalent with :

$$\left((a+b+c)(a^2+b^2+c^2) - 9abc\right)\left((a-b)^2+(b-c)^2+(c-a)^2\right) \geq 0$$